Calculus Section 3.4 Concavity and the Second Derivative Test

-Determine intervals on which a function is concave upward or concave downward

-Find any points of inflection of the graph of a function

-Apply the Second Derivative test to find relative extrema of a function

Homework: page 192 #'s 1, 2, 15, 17, 24, 33, 34, 50, 77, 78

Definition of Concavity

Let f be differentiable on an open interval. The graph of f is ___concave up___ if f' is increasing on the interval and ___concave down___ if f' is decreasing on the interval.

slopes getting less negative, more positive

slopes getting less positive, more negative
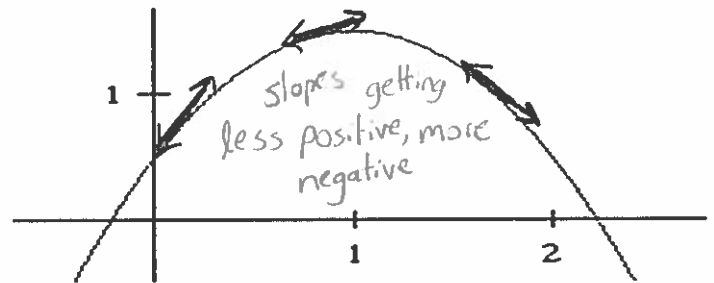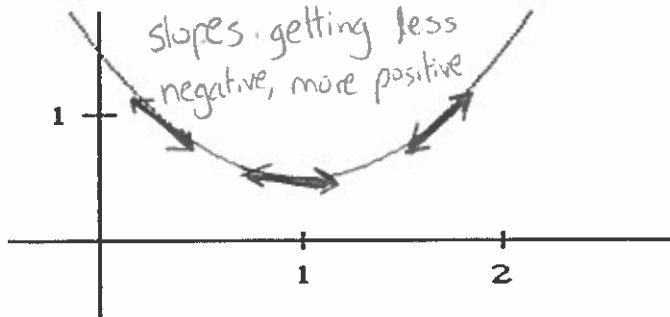
Test for Concavity

Let f be a function whose second derivative exists on an open interval I.

1) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward in I.

2) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward in I.

Example)

Determine the open intervals on which the graph of $f(x) = x^4 - 4x^3 + 2$ is concave upward or downward.

$f'(x) = 4x^3 - 12x^2$

$f''(x) = 12x^2 - 24x$

$0 = 12x(x-2)$

$x=0 \quad x=2$

x	-1	0	1	2	3
$f''(x)$	+	0	-	0	+

concave up: $(-\infty, 0) \cup (2, \infty)$

concave down: $(0, 2)$

Definition of Points of Inflection

Let f be a function that is continuous on an open interval and let c be a point in the interval. If the graph of f ___changes concavity___ at that point, then this point is a **point of inflection** of the graph of f.

Points of Inflection

If (c, f(c)) is a point of inflection of the graph of f, then either $f''(c) = 0$ or f'' does not exist at x = c.

Example: Finding points of inflection

Determine the points of inflection and discuss the concavity of the graph of $f(x) = x^{\frac{2}{3}}(x-5)$.

$f(x) = x^{5/3} - 5x^{2/3}$

$f'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3}$

$f''(x) = \frac{10}{9}x^{-1/3} + \frac{10}{9}x^{-4/3}$

$0 = \frac{10}{9x^{1/3}} + \frac{10}{9x^{4/3}}$

$\frac{-10}{9x^{1/3}} = \frac{10}{9x^{4/3}}$ ✓ DNE at x=0

$-90x^{4/3} = 90x^{1/3}$

$-x^{4/3} = x^{1/3}$

$-x = 1$

$x = -1$

x	-2	-1	-½	0	1
f''(x)	−	0	+	DNE	+

concave up: $(-1, 0) \cup (0, \infty)$

concave down: $(-\infty, -1)$

Point of inflection: x = -1

The Second Derivative Test

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1) If $f''(c) > 0$ (concave up), then f has a ___relative maximum___ at (c, f(c)).

2) If $f''(c) < 0$ (concave down), then f has a ___relative minimum___ at (c, f(c)).

3) If $f''(c) = 0$, then ___the test is inconclusive___. It tells you NOTHING. You have to use the First Derivative Test instead. This does not mean that there is no max/min, the test just doesn't work for that function.

Example)

Use a tangent line at x = 1 to approximate f(1.1) for the function $f(x) = 3x^2 + 2$. Tell whether the function is an overestimate or an underestimate. Justify your answer.

$f'(x) = 6x$

$f''(x) = 6$

$f'(1) = 6 \qquad f(1) = 5$

$y - 5 = 6(x-1)$

$y = 6(1.1-1) + 5$

$y = 6(.1) + 5$

$y = 5.6$

$f(1.1) \approx 5.6$

$f(1.1) \approx 5.6$ is an overestimate because $f''(x) > 0$

inc/concave up — underestimate

inc/concave down — overestimate

dec/concave up — underestimate

dec/concave down — overestimate